On P_S-Compact Spaces

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Abstract—In this paper, we introduce a new class of spaces namely P_S-compact. This class of compactness lies strictly between the classes of strongly compact and nearly compact spaces, but it is not comparable with compact space.

Index Terms—P_S-open sets, P_S-continuous functions, P_S-compact spaces, strongly compact spaces.

1 Introduction

Mashhour et al. [14] and Levine [12] defined preopen and semi-open sets, respectively, which are both weaker than open sets in topological spaces. In 2009, Khalaf and Asaad [11] introduced P_S-open sets, which are stronger than preopen sets, in order to investigate the characterization of P_S-continuous functions. In [10] they have introduced the notion of almost P_S-continuous functions. Singal and Mathur [21] defined the concept of nearly compact spaces. Mashhour et al. [15] introduced the concept of strongly compact spaces. The purpose of the present paper is to introduce a new class of spaces called P_S-compact. This class of spaces lies strictly between the classes of strongly compact space and nearly compact space, but it is not comparable with compact space.

2 Preliminaries

Throughout this paper, (X,τ) and (Y,σ) (or simply X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. If A is any subset of a space X, then Cl(A) and Int(A) denote the closure and the interior of A, respectively. A subset A of X is called preopen [14] (resp., semi-open [12], α-open [18] and regular open [22]) if $A \subseteq Int(Cl(A))$ (resp., $A \subseteq Cl(Int(A))$, $A \subseteq Int(Cl(Int(A)))$ and $A = Int(Cl(A))$). The complement of a preopen (resp., semi-open) set is called preclosed (resp., semi-closed). A subset A of X is said to be preregular [17] if A is both preopen and preclosed. A preopen subset A of X is called P_S-open [11] if for each $x \in A$, there exists a semi-closed set F such that $x \in F \subseteq A$. The complement of a P_S-open set is called P_S-closed. The intersection (resp., union) of an arbitrary collection of P_S-closed (resp., P_S-open) sets in (X,τ) is P_S-closed (resp., P_S-open). A subset A of a space X is called θ-semi-open [7] if for each $x \in A$, there exists a semi-open set G such that $x \in G \subseteq Cl(G) \subseteq A$.

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The P_S-closure (resp., preclosure and semi-closure) of A, denoted by $P_SCl(A)$ (resp., pCl(A) and sCl(A)), is defined as the intersection of all P_S-closed (resp., preclosed and semi-closed) sets containing A. The semi-interior of A, denoted by sInt(A), is defined as the union of all semi-open sets contained in A. A subset A of X is called regular semi-open [2] if A = sInt(sCl(A)). The complement of a regular semi-open) set is called preclosed (resp., semi-closed and regular semi-open). A point $x \in X$ is called a δ-cluster [23] of A if $A \cap U \neq \phi$ for each regular open set U containing x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by $Cl_\delta(A)$. A subset A is called δ-closed if $Cl_\delta(A) = A$. The complement of a δ-closed set is called δ-open. We denote the collection of all P_S-open (resp., preopen, regular open and regular semi-open) sets of X by $P_SO(X)$ (resp., PO(X), RO(X) and RSO(X)).

Recall that a space X is said to be extremally disconnected [5] if $Cl(U) \in \tau$ for every $U \in \tau$. A space X is called locally indiscrete [3] if every open subset of X is closed. A space X is said to be hyperconnected [3] if every nonempty open subset of X is dense in X. A space X is s-regular [1] if and only if for each $x \in X$ and each open set G containing x, there exists a semi-open set H such that $x \in H \subseteq sCl(H) \subseteq G$. A space X is called semi-T_1 [13] if and only if to each pair of distinct points x, y of X, there exists a pair of semi-open sets, one containing x but not y and the other containing y but not x. A function $f : X \rightarrow Y$ is said to be P_S-continuous [11] (resp., precontinuous [14] and θs-continuous [9]) if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a P_S-open (resp., preopen and θ-semi-open) set U of X containing x such that $f(U) \subseteq V$. A function $f : X \rightarrow Y$ is said to be almost P_S-continuous [10] (resp., almost precontinuous [16] and almost θs-continuous [8]) if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a P_S-open (resp., preopen and θ-semi-open) set U of X containing x such that $f(U) \subseteq Int(Cl(V))$

Recall that a filter base $\Im$ is said to be p-converges [6] (resp., pre-θ-converges [4] and δ-converges [23]) to a point $x \in X$ if for every preopen (resp., preopen and open) set V containing x, there exists an $F \in \Im$ such that $F \subseteq V$ (resp., $F \subseteq pCl(V)$ and $F \subseteq Int(Cl(V))$). A filter base $\Im$ is said to be p-accumulates [6] (resp., pre-θ-accumulates [4] and δ-accumulates [23]) to a

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point $x \in X$ if $F \cap V \neq \phi$ (resp., $F \cap pCl(V) \neq \phi$ and $F \cap Int(Cl(V)) \neq \phi$), for every preopen (resp., preopen and open) set V containing x and every $F \in \Im$. A topological space (X,τ) is said to be strongly compact [15] (resp., α-compact [4]) if every preopen (resp., α-open) cover of X has a finite subcover. A subset A of a space X is said to be N-closed [19] (resp., quasi-H-closed [20]) relative to X if for every cover $\{V_\alpha : \alpha \in \Delta\}$ of A by open sets of X, there exists a finite subset Δ_0 of Δ such that $A \subseteq \cup\{Int(Cl(V_\alpha)) : \alpha \in \Delta_0\}$ (resp., $A \subseteq \cup\{Cl(V_\alpha) : \alpha \in \Delta_0\}$). A space X is said to be nearly compact [21] if X is N-closed relative to X.

Lemma 2.1 [11]: The following statements are true:

1) If a space X is semi-T_1, then $P_sO(X) = PO(X)$.
2) If a space X is locally indiscrete, then $P_sO(X) = \tau$.
3) If a space X is s-regular, then $\tau \subseteq P_sO(X)$.
4) A space X is hyperconnected if and only if $P_sO(X) = \{\phi, X\}$.

Lemma 2.2 [11]: Let (Y,τ_Y) be a subspace of a space (X,τ) and $A \subseteq X$. Then the following properties are hold:

1) If $A \in P_sO(Y)$ and $Y \in RO(X)$, then $A \in P_sO(X)$.
2) If $A \in P_sO(X)$ and $Y \in RO(X)$, then $A \cap Y \in P_sO(X)$.
3) If either $Y \in RSO(X)$ or $Y \in \tau$ or Y is a preregular, and $A \in P_sO(X)$, then $A \cap Y \in P_sO(Y)$.
4) If $Y \in RO(X)$, then $P_sO(Y) = P_sO(X) \cap Y$.

Theorem 2.3 [11]: If $f : X \to Y$ is a continuous and open function and V is a P_s-open set of Y, then $f^{-1}(V)$ is a P_s-open set of X.

Theorem 2.4 [11]: Let $f : X \to Y$ be a function and X be an extremally disconnected space. If f is θs-continuous (resp., almost θs-continuous), then f is P_s-continuous (resp., almost P_s-continuous).

3 P_S-Compact Spaces

In this section, we introduce a new class of topological spaces called P_s-compact. This class of spaces lies strictly between the classes of strongly compact space and nearly compact space, but it is not comparable with compact space.

Definition 3.1: A filter base $\Im$ in a topological space (X,τ) P_s-converges (resp., P_s-θ-converges) to a point $x \in X$ if for every P_s-open set V containing x, there exists an $F \in \Im$ such that $F \subseteq V$ (resp., $F \subseteq P_sCl(V)$).

Definition 3.2: A filter base $\Im$ in a topological space (X,τ) P_s-accumulates (resp., P_s-θ-accumulates) to a point $x \in X$ if $F \cap V \neq \phi$ (resp., $F \cap P_sCl(V) \neq \phi$), for every P_s-open set V containing x and every $F \in \Im$.

It is clear from the above definitions that P_s-converges (resp., P_s-accumulates) of filter bases in topological spaces implies P_s-θ-converges (resp., P_s-θ-accumulates), but the converses are not true in general as shown in the following example.

Example 3.3: Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\Im = \{\{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}$. Then $P_sO(X) = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Thus, $\Im$ P_s-θ-converges to a, but $\Im$ does not P_s-converges to a, because the set $\{a\}$ is P_s-open containing a, there is no an $F \in \Im$ such that $F \subseteq \{a\}$. Also $\Im$ P_s-θ-accumulates to b, but $\Im$ does not P_s-accumulates to b, because the set $\{b\}$ is P_s-open containing b, there exists an $F \in \Im$ such that $F \cap \{b\} = \phi$.

The following proposition is an easy consequence of the above definitions.

Proposition 3.4: If $\Im$ is a maximal filter base in a topological space (X,τ), then $\Im$ P_s-converges (resp., P_s-θ-converges) to a point $x \in X$ if and only if $\Im$ P_s-accumulates (resp., P_s-θ-accumulates) to a point x.

Lemma 3.5: Let $\Im$ be a filter base in a topological space (X,τ). If $\Im$ p-converges (resp., pre-θ-converges) to a point $x \in X$, then $\Im$ P_s-converges (resp., P_s-θ-converges) to a point x.

Proof: Suppose that $\Im$ p-converges (resp., pre-θ-converges) to a point $x \in X$. Let V be any P_s-open set containing x, then V is preopen set containing x. Since $\Im$ p-converges (resp., pre-θ-converges) to a point $x \in X$, there exists an $F \in \Im$ such that $F \subseteq V$ (resp., $F \subseteq pCl(V) \subseteq P_sCl(V)$). This shows that $\Im$ P_s-converges (resp., P_s-θ-converges) to a point x.

The following example shows that the converse of Lemma 3.5 is not true in general.

Example 3.6: Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{d\}, \{b, c\}, \{b, c, d\}, \{a, b, c\}\}$ and $\Im = \{\{a, c\}, \{a, c, d\}, \{a, b, c\}, X\}$. Then $P_sO(X) = \{\phi, X, \{d\}, \{a, b, c\}\}$. Thus, $\Im$ P_s-converges (resp., P_s-θ-converges) to a, but $\Im$ does not pre-θ-converges to a and hence does not p-converges to a, because the set $\{a, b\}$ is preopen containing a, there is no an $F \in \Im$ such that $F \subseteq \{a, b\}$ (resp., $F \subseteq pCl(\{a, b\}) = \{a, b\}$).

Lemma 3.7: Let $\Im$ be a filter base in a topological space (X,τ). If $\Im$ p-accumulates (resp., pre-θ-accumulates) to a point $x \in X$, then $\Im$ P_s-accumulates (resp., P_s-θ-accumulates) at a point x.

Proof: The proof is similar to Lemma 3.5.

The converse of Lemma 3.7 is not true in general as shown by the following example.

Example 3.8: Consider the space (X,τ) given in Example 3.6. Let $\Im = \{\{a, c\}, \{a, c, d\}, \{a, b, c\}, X\}$. Then $\Im$ P_s-accumulates (resp., P_s-θ-accumulates) to b, but $\Im$ does not pre-θ-accumulates to b and hence does not p-accumulates to b, because the set $\{b, d\}$ is preopen containing b, there exists an $F \in \Im$ such that $F \cap pCl(\{b, d\}) = \phi$ and hence $F \cap \{b, d\} = \phi$.

Lemma 3.9: Let $\Im$ be a filter base in a topological space (X,τ). If $\Im$ P_s-converges to a point $x \in X$, then $\Im$ δ-converges to a point x.

Proof: Suppose that $\Im$ P_s-converges to a point $x \in X$. Let V be any open set containing x, then $Int(Cl(V)) \in RO(X)$. Since

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$RO(X) \subseteq P_SO(X)$ in general, so $Int(Cl(V)) \in P_SO(X)$. Since $\Im$ P_S-converges to a point $x \in X$, there exists an $F \in \Im$ such that $F \subseteq Int(Cl(V))$. This shows that $\Im$ δ-converges at a point x.

Lemma 3.10: Let $\Im$ be a filter base in a topological space (X,τ). If $\Im$ P_S-accumulates to a point $x \in X$, then $\Im$ δ-accumulates to x.
Proof: The proof is similar to Lemma 3.9.

Proposition 3.11: Let $\Im$ be a filter base in a topological space (X,τ) and E is any semi-closed set containing x. If there exists an $F \in \Im$ such that $F \subseteq E$ (resp., $F \subseteq P_SCl(E)$), then $\Im$ P_S-converges (resp., P_S-θ-converges) to a point $x \in X$.
Proof: Let V be any P_S-open set containing x, then for each $x \in V$, there exists a semi-closed set E such that $x \in E \subseteq V$. By hypothesis, there exists an $F \in \Im$ such that $F \subseteq E \subseteq V$ (resp., $F \subseteq P_SCl(E) \subseteq P_SCl(V)$) which implies that $F \subseteq V$ (resp., $F \subseteq P_SCl(V)$). Hence $\Im$ P_S-converges (resp., P_S-θ-converges) to a point $x \in X$.

Proposition 3.12: Let $\Im$ be a filter base in a topological space (X,τ) and E is any semi-closed set containing x. If there exists an $F \in \Im$ such that $F \cap E \neq \phi$ (resp., $F \cap P_SCl(E) \neq \phi$), then $\Im$ is P_S-accumulation (resp., P_S-θ-accumulation) to a point $x \in X$.
Proof: The proof is similar to Proposition 3.11.

Theorem 3.13: If a function $f : X \rightarrow Y$ is P_S-continuous (resp., almost P_S-continuous), then for each point $x \in X$ and each filter base $\Im$ in X P_S-converging to x, the filter base $f(\Im)$ is convergent (resp., δ-convergent) to $f(x)$. Furthermore, if X is submaximal, then the converse also holds.
Proof: Suppose that x belongs to X and $\Im$ is any filter base in X which P_S-converges to x. By the P_S-continuity (resp., almost P_S-continuity) of f, for any open set V in Y containing $f(x)$, there exists $U \in P_SO(X)$ containing x such that $f(U) \subseteq V$ (resp., $f(U) \subseteq Int(Cl(V))$). But $\Im$ is P_S-convergent to x in X, then there exists an $F \in \Im$ such that $F \subseteq U$. It follows that $f(F) \subseteq V$ (resp., $f(F) \subseteq Int(Cl(V))$). This means that $f(\Im)$ is convergent (resp., δ-convergent) to $f(x)$.
Now suppose that X is submaximal. Let x be a point in X and V any open set containing $f(x)$. Since X is submaximal, then every P_S-open set of X is open in X. If we set $\Im = P_SO(X)$ containing x, then $\Im$ will be a filter base which P_S-converges to x. So there exists U in $\Im$ such that $f(U) \subseteq V$ (resp., $f(U) \subseteq Int(Cl(V))$). Therefore, f is P_S-continuous (resp., almost P_S-continuous).
Definition 3.14: We say that a topological space (X,τ) is P_S-compact if for every P_S-open cover $\{V_\alpha: \alpha \in \Delta\}$ of X, there exists a finite subset Δ_0 of Δ such that $X = \cup\{V_\alpha: \alpha \in \Delta_0\}$.

Theorem 3.15: If every semi-closed cover of a space X has a finite subcover, then X is P_S-compact.
Proof: Let $\{V_\alpha: \alpha \in \Delta\}$ be any P_S-open cover of X, then for each $x \in X$, there exists $\alpha \in \Delta_0$, $x \in V_{\alpha(x)}$, there exists a semi-closed set $F_{\alpha(x)}$ such that $x \in F_{\alpha(x)} \subseteq V_{\alpha(x)}$. So the family $\{F_{\alpha(x)}: x \in X\}$ is a cover of X by semi-closed set, then by hypothesis, this family has a finite subcover such that $X = \{F_{\alpha(xi)}: i = 1, 2, \ldots, n\} \subseteq \{V_{\alpha(xi)}: i = 1, 2, \ldots, n\}$. Therefore, $X = \{V_{\alpha(xi)}: i = 1, 2, \ldots, n\}$. Hence X is P_S-compact.

The following lemma shows the relation between strongly compact and P_S-compact spaces.
Lemma 3.16: If a topological space (X,τ) is strongly compact, then it is P_S-compact.
Proof: Let $\{V_\alpha: \alpha \in \Delta\}$ be any P_S-open cover of X. Then $\{V_\alpha: \alpha \in \Delta\}$ is a preopen cover of X. Since X is strongly compact, there exists a finite subset Δ_0 of Δ such that $X = \cup\{V_\alpha: \alpha \in \Delta_0\}$. Hence X is P_S-compact.

The converse of Lemma 3.16 is not true as shown by the next example.
Example 3.17: Let $X = \Re$ with the topology $\tau = \{\phi, X, \{0\}\}$. Then (X,τ) is not strongly compact [4, Example 2.6 (iii)]. Since the space X is hyperconnected, then by Lemma 2.1 (4), $P_SO(X) = \{\phi, X\}$. Then (X,τ) is P_S-compact.

Theorem 3.18: Every semi-T_1 and P_S-compact space is strongly compact.
Proof: Suppose that X is semi-T_1 and P_S-compact space. Let $\{V_\alpha: \alpha \in \Delta\}$ be any preopen cover of X. Then for every $x \in X$, there exists $\alpha(x) \in \Delta$ such that $x \in V_{\alpha(x)}$. Since X is semi-T_1, by Lemma 2.1 (1), the family $\{V_\alpha: \alpha \in \Delta\}$ is a P_S-open cover of X. Since X is P_S-compact, there exists a finite subset Δ_0 of Δ in X such that $X = \cup\{V_\alpha: \alpha \in \Delta_0\}$. Hence X is strongly compact.
Corollary 3.19: Let a space X be semi-T_1. Then X is P_S-compact if and only if X is strongly compact.
Proof: Follows from Lemma 3.16 and Theorem 3.18.
Lemma 3.20: If a topological space (X,τ) is P_S-compact, then it is nearly compact.
Proof: Let $\{V_\alpha: \alpha \in \Delta\}$ be any regular open cover of X. Then $\{V_\alpha: \alpha \in \Delta\}$ is a P_S-open cover of X. Since X is P_S-compact, there exists a finite subset Δ_0 of Δ such that $X = \cup\{V_\alpha: \alpha \in \Delta_0\}$. Hence X is nearly compact.

The following example shows that the converse of the above lemma is not true.
Example 3.21: The unit interval [0,1] with the usual topology is compact [4, Example 2.6 (ii)] and hence it is nearly compact, but not P_S-compact.
Corollary 3.22: If a topological space (X,τ) is P_S-compact, then it is quasi-H-closed.
Proof: Follows from Lemma 3.20 and the fact that each nearly compact space is quasi-H-closed.

In general, P_S-compact spaces and compact spaces are not comparable as shown by the following two examples:

Example 3.23: Let $X = (0, 1)$ with the topology τ consisting of ϕ, X and all subsets of X of the form $(0,1-1/n)$, where $n = 2, 3, \ldots$. Then (X,τ) is not compact (see [22]). Since the space X is hyperconnected, then by Lemma 2.1 (4), the family of P_s-open sets are only ϕ and X. Therefore, X is P_s-compact.

Example 3.24: Any closed interval $[a,b]$, where $a, b \in \Re$ with the relative usual topology is compact, but it is not P_s-compact.

Proposition 3.25: Let a topological space (X,τ) is locally indiscrete. Then X is compact if and only if X is P_s-compact.

Proof: Follows from Lemma 2.1 (2).

Lemma 3.26: Let a topological space (X,τ) be s-regular. If X is P_s-compact, then it is compact.

Proof: Follows from Lemma 2.1 (3).

From Lemma 3.16, Lemma 3.20 and Corollary 3.22, we obtain the following diagram of implications.

$$\begin{tikzcd}
\textbf{strongly compact} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \textbf{P}_s\textbf{-compact} \arrow[r, Rightarrow] & \textbf{nearly compact} \arrow[d, Rightarrow] \\
\boldsymbol{\alpha}\textbf{-compact} \arrow[r, Rightarrow] & \textbf{compact} \arrow[r, Rightarrow] \arrow[ur] & \textbf{quasi-H-closed}
\end{tikzcd}$$

Diagram 3.1

Theorem 3.27: For any topological space (X,τ). The following statements are equivalent:

1) (X,τ) is P_s-compact,

2) Every maximal filter base $\Im$ in X P_s-converges to some point $x \in X$,

3) Every filter base $\Im$ in X P_s-accumulates to some point $x \in X$,

4) For every family $\{F_\alpha: \alpha \in \Delta\}$ of P_s-closed subsets of X such that $\cap\{F_\alpha: \alpha \in \Delta\} = \phi$, there exists a finite subset Δ_0 of Δ such that $\cap\{F_\alpha: \alpha \in \Delta_0\} = \phi$.

Proof: (1) ⇒ (2): Suppose that X is P_s-compact space and let $\Im = \{F_\alpha: \alpha \in \Delta\}$ be a maximal filter base. Suppose that $\Im$ does not P_s-converges to any point of X. Since $\Im$ is maximal, by Proposition 3.4, $\Im$ does not P_s-accumulates to any point of X. This implies that for every $x \in X$, there exists a P_s-open set V_x and an $F_{\alpha(x)} \in \Im$ such that $F_{\alpha(x)} \cap V_x = \phi$. The family $\{V_x: x \in X\}$ is a P_s-open cover of X and by hypothesis, there exists a finite number of points $x_1, x_2, \ldots, x_n$ of X such that $X = \cup\{V_{(xi)}: i = 1, 2, \ldots, n\}$. Since $\Im$ is a filter base on X, there exists an $F_0 \in \Im$ such that $F_0 \subseteq \cap\{F_{\alpha(xi)}: i = 1, 2, \ldots, n\}$. Hence $F_0 \cap V_{(xi)} = \phi$ for $i = 1, 2, \ldots, n$. Which implies that $F_0 \cap \{\cup V_{(xi)}: i = 1, 2, \ldots, n\} = F_0 \cap X = \phi$. Therefore, we obtain $F_0 = \phi$. Contracting the fact that $F_0 \neq \phi$.

(2) ⇒ (3): Let $\Im$ be any filter base on X. Then, there exists a maximal filter base $\Im_0$ such that $\Im \subseteq \Im_0$. By hypothesis, $\Im_0$ P_s-converges to some point $x \in X$. For every $F \in \Im$ and every P_s-open set V containing x, there exists an $F_0 \in \Im_0$ such that $F_0 \subseteq V$, hence $\phi \neq F_0 \cap F \subseteq V \cap F$. This shows that $\Im$ P_s-accumulates at x.

(3) ⇒ (4): Let $\{F_\alpha: \alpha \in \Delta\}$ be a family of P_s-closed subsets of X such that $\cap\{F_\alpha: \alpha \in \Delta\} = \phi$. Suppose that every finite subfamily $\cap\{F_{\alpha i}: i = 1, 2, \ldots, n\} \neq \phi$. Therefore $\Im = \{\cap F_{\alpha i}: i = 1, 2, \ldots, n, F_{\alpha i} \in \{F_\alpha: \alpha \in \Delta\}\}$ form a filter base on X. By hypothesis, $\Im$ P_s-accumulates to some point $x \in X$. This implies that for every P_s-open set V containing x, $F_\alpha \cap V \neq \phi$ for every $F_\alpha \in \Im$ and every $\alpha \in \Delta$. Since $x \notin \cap F_\alpha$ there exists $\alpha_0 \in \Delta$ such that $x \notin F_{\alpha 0}$. Hence, x is contained in the P_s-open set $X \backslash F_{\alpha 0}$ and $F_{\alpha 0} \cap X \backslash F_{\alpha 0} = \phi$. Contracting the fact that $\Im$ P_s-accumulates to x.

(4) ⇒ (1): Let $\{V_\alpha: \alpha \in \Delta\}$ be a P_s-open cover of X. Then $\{X \backslash V_\alpha: \alpha \in \Delta\}$ is a family of P_s-closed subsets of X such that $\cap\{X \backslash V_\alpha: \alpha \in \Delta\} = \phi$. By hypothesis, there exists a finite subset Δ_0 of Δ such that $\cap\{X \backslash V_\alpha: \alpha \in \Delta_0\} = \phi$. Hence $X = \cup\{V_\alpha: \alpha \in \Delta_0\}$. This shows that X is P_s-compact.

4 P_S-SETS AND P_S-COMPACT SUBSPACES

In this section, we introduce a new class of topological space called P_s-set and P_s-compact subspace.

Definition 4.1: A subset A of a topological space (X,τ) is said to be P_s-set (resp., P_s-compact subspace) if for every cover $\{V_\alpha: \alpha \in \Delta\}$ of A by P_s-open subsets of (X,τ) (resp., by P_s-open subsets of A), there exists a finite subset Δ_0 of Δ such that $A \subseteq \cup\{V_\alpha: \alpha \in \Delta_0\}$ (resp., $A = \cup\{V_\alpha: \alpha \in \Delta_0\}$).

Lemma 4.2: A subset A of a space X is P_s-set (resp., P_s-compact subspace) if and only if for every cover of A by P_s-open sets of X (resp., by P_s-open sets of A) has a finite subcover.

Proof: The proof follows directly from Definition 4.1.

Now we will give several equivalent conditions to P_s-sets (resp., P_s-compact subspaces) of topological spaces and also we give some other conditions each of which makes a given topological space a P_s-compact space.

Theorem 4.3: Let A be a subset of a topological space (X,τ). If every cover of A by semi-closed subsets of X (resp., by semi-closed subsets of A) has a finite subcover, then A is P_s-set (resp., P_s-compact subspace).

Proof: Let $\{V_\alpha: \alpha \in \Delta\}$ be a cover of A by P_s-open subset of X (resp., by P_s-open subsets of A), then for each $x \in X$, there exists $\alpha \in \Delta_0$, $x \in V_{\alpha(x)}$, there exists a semi-closed set $F_{\alpha(x)}$ such that $x \in F_{\alpha(x)} \subseteq V_{\alpha(x)}$. So the family $\{F_{\alpha(x)}: x \in X\}$ is a cover of A by semi-closed subsets of X (resp., by semi-closed subsets of A), then by hypothesis, this family has a finite subcover such that $A \subseteq \cup\{F_{\alpha(xi)}: i = 1, 2, \ldots, n\} \subseteq \cup\{V_{\alpha(xi)}: i = 1, 2, \ldots, n\}$ (resp., $A = \cup\{F_{\alpha(xi)}: i = 1, 2, \ldots, n\} \subseteq \cup\{V_{\alpha(xi)}: i = 1, 2, \ldots, n\}$. Therefore, $A \subseteq \cup\{V_{\alpha(xi)}: i = 1, 2, \ldots, n\}$ (resp., $A = \cup\{V_{\alpha(xi)}: i = 1, 2, \ldots, n\}$. Hence A is P_s-set (resp., P_s-compact subspace).

Theorem 4.4: For any topological space (X,τ). The following statements are equivalent:

1) A is P_s-set (resp., P_s-compact subspace),

2) Every maximal filter base $\Im$ on X which meets A P_s-converges to some point of A,

3) Every filter base $\Im$ on X which meets A P_s-accumulates to some point $x \in X$,

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4) For every family $\{F_\alpha: \alpha \in \Delta\}$ of P_S-closed subsets of (X,τ) such that $[\cap\{F_\alpha: \alpha \in \Delta\}] \cap A = \phi$, there exists a finite subset Δ_0 of Δ such that $[\cap\{F_\alpha: \alpha \in \Delta_0\}] \cap A = \phi$.

Proof: Similar to Theorem 3.27.

Theorem 4.5: A space X is P_S-compact if and only if every proper P_S-closed set of X is P_S-set.

Proof: Necessity: Let F be any proper P_S-closed set of X. Let $\{V_\alpha: \alpha \in \Delta\}$ be a cover of F and $V_\alpha \in P_SO(X)$ for every $\alpha \in \Delta$. Since F is P_S-closed set, then $X\backslash F$ is P_S-open set. So the family $\{V_\alpha: \alpha \in \Delta\} \cup X\backslash F$ is a P_S-open cover of X. Since X is P_S-compact, there exists a finite subset Δ_0 of Δ such that $X = \cup\{V_\alpha: \alpha \in \Delta_0\} \cup (X\backslash F)$. Therefore, we obtain $F \subseteq \cup\{V_\alpha: \alpha \in \Delta_0\}$. Hence F is P_S-set.

Sufficiency: Let $\{V_\alpha: \alpha \in \Delta\}$ be a cover of X and $V_\alpha \in P_SO(X)$ for every $\alpha \in \Delta$. Suppose that $X \neq V_{\alpha 0} \neq \phi$ for every $\alpha 0 \in \Delta$. Then $X\backslash V_{\alpha 0}$ is a proper P_S-closed subset of X. Therefore, by hypothesis, there exists a finite subset Δ_0 of Δ such that $X\backslash V_{\alpha 0} \subseteq \cup\{V_\alpha: \alpha \in \Delta_0\}$. Therefore, we obtain $X = \cup\{V_\alpha: \alpha \in \Delta_0 \cup \{\alpha 0\}\}$. Which shows that X is P_S-compact.

Theorem 4.6: If a space X is P_S-compact and A is both regular open and P_S-closed subset of X, then A is P_S-compact subspace.

Proof: Let $\{A_\alpha: \alpha \in \Delta\}$ be any cover of A by P_S-open set of A. Since $A \in RO(X)$, by Lemma 2.2 (1), $A_\alpha \in P_SO(X)$ for each $\alpha \in \Delta$. Since A is a P_S-closed subset of X, then $X\backslash A \in P_SO(X)$ and $\{A_\alpha: \alpha \in \Delta\} \cup X\backslash A = X$ and $\{A_\alpha: \alpha \in \Delta\} \cup X\backslash A$ forms a P_S-open cover of X. Since X is P_S-compact, there exists a finite subset Δ_0 of Δ such that $X = \cup\{A_\alpha: \alpha \in \Delta_0\} \cup X\backslash A$, hence $A = \cup\{A_\alpha: \alpha \in \Delta_0\}$. Therefore, A is P_S-compact subspace.

Theorem 4.7: If there exists either a proper regular semi-open or a proper preregular subset A of a topological space (X,τ) such that A and $X\backslash A$ are P_S-compact subspace, then X is also P_S-compact.

Proof: Let $\{V_\alpha: \alpha \in \Delta\}$ be any P_S-cover cover of X. Since A is either regular semi-open or preregular subset of X, then for every $\alpha \in \Delta$, by Lemma 2.2 (3), we have $A \cap V_\alpha \in P_SO(A)$. Therefore, $\{A \cap V_\alpha: \alpha \in \Delta\}$ is a P_S-open cover of A. Since A is P_S-compact subspace, there exists a finite subset Δ_0 of Δ such that $A = \cup\{A \cap V_\alpha: \alpha \in \Delta_0\}$. Therefore, we have $A \subseteq \cup\{V_\alpha: \alpha \in \Delta_0\}$. Since A is either regular semi-open or preregular subset of X, then $X\backslash A$ is also either regular semi-open or preregular. By the same way we can find a finite subset Δ_1 of Δ such that $X\backslash A \subseteq \cup\{V_\alpha: \alpha \in \Delta_1\}$. Hence $X = \cup\{V_\alpha: \alpha \in \Delta_0 \cup \Delta_1\}$. This shows that X is P_S-compact.

Theorem 4.8: If there exists a proper clopen subset A of a topological space (X,τ) such that A and $X\backslash A$ are P_S-sets, then X is also P_S-compact.

Proof: Let $\{V_\alpha: \alpha \in \Delta\}$ be any P_S-cover cover of X. Since A is clopen subset of X, then A is regular open subset of X. Therefore, for every $\alpha \in \Delta$, by Lemma 2.2 (2), we have $A \cap V_\alpha \in P_SO(X)$. Therefore, $\{A \cap V_\alpha: \alpha \in \Delta\}$ is a cover of A by P_S-open sets of X. Since A is P_S-set, there exists a finite subset Δ_0 of Δ such that $A \subseteq \cup\{V_\alpha: \alpha \in \Delta_0\} \cap A \subseteq \cup\{V_\alpha: \alpha \in \Delta_0\}$. Since A is clopen subset of X, then $X\backslash A$ is also clopen. By the same way we can find a finite subset Δ_1 of Δ such that $X\backslash A \subseteq \cup\{V_\alpha: \alpha \in \Delta_1\}$. Hence $X = \cup\{V_\alpha: \alpha \in \Delta_0 \cup \Delta_1\}$. This shows that X is P_S-compact.

Theorem 4.9: If a regular open set G of a space X is P_S-set, then G is P_S-compact subspace.

Proof: Suppose that $G \in RO(X)$ and G is P_S-set. Let $\{V_\alpha: \alpha \in \Delta\}$ be a cover of G and $V_\alpha \in P_SO(G)$ for every $\alpha \in \Delta$. Since $G \in RO(X)$, then by Lemma 2.2 (1), we have $V_\alpha \in P_SO(X)$ for every $\alpha \in \Delta$. Since G is P_S-set, there exists a finite subset Δ_0 of Δ such that $G \subseteq \cup\{V_\alpha: \alpha \in \Delta_0\}$, which implies that G is P_S-compact subspace.

Theorem 4.10: If either $G \in RSO(X)$ or $G \in \tau$ or G is a preregular or an open set G of a space X is P_S-compact subspace, then G is P_S-set.

Proof: Suppose that either $G \in RSO(X)$ or $G \in \tau$ or G is a preregular, and $\{V_\alpha: \alpha \in \Delta\}$ be a cover of G and $V_\alpha \in P_SO(X)$ for every $\alpha \in \Delta$. Since either $G \in RSO(X)$ or $G \in \tau$ or G is a preregular, then for every $\alpha \in \Delta$, by Lemma 2.2 (3), we have $G \cap V_\alpha \in P_SO(G)$. Therefore, the family $\{G \cap V_\alpha: \alpha \in \Delta\}$ is a P_S-open cover of G. Since G is P_S-compact subspace, there exists a finite subset Δ_0 of Δ such that $G = \cup\{G \cap V_\alpha: \alpha \in \Delta_0\}$. Therefore, $G \subseteq \cup\{V_\alpha: \alpha \in \Delta_0\}$, which implies that G is P_S-set.

Corollary 4.11: A regular open set G of a space X is P_S-set if and only if G is P_S-compact subspace.

Proof: This is an immediate consequence of Theorem 4.9 and Theorem 4.10.

Theorem 4.12: Let A and B be subsets of a space X. If A is P_S-closed set and B is P_S-set, then $A \cap B$ is P_S-set.

Proof: Let $\{V_\alpha: \alpha \in \Delta\}$ be any cover of $A \cap B$ by P_S-open subsets of X. Since A is P_S-closed set, then $X\backslash A$ is P_S-open. So $B \subseteq \cup\{V_\alpha: \alpha \in \Delta\} \cup X\backslash A$ and the family $\{V_\alpha: \alpha \in \Delta\} \cup X\backslash A$ is a P_S-open cover of B. Since B is P_S-set, then there exists a finite subset Δ_0 of Δ such that $B \subseteq \cup\{V_\alpha: \alpha \in \Delta_0\} \cup (X\backslash A)$. Therefore, we obtain $A \cap B \subseteq \cup\{V_\alpha: \alpha \in \Delta_0\}$. Hence $A \cap B$ is P_S-set.

Theorem 4.13: Let Y be any regular open subspace of a space X and A be any subset of Y. Then A is P_S-set of X if and only if A is P_S-set of Y.

Proof: Necessity: Suppose that A is P_S-set of X and $Y \in RO(X)$. Let $\{V_\alpha: \alpha \in \Delta\}$ be a cover of A and $V_\alpha \in P_SO(Y)$ for every $\alpha \in \Delta$. Since $Y \in RO(X)$. Then by Lemma 2.2 (4), there exists a P_S-open set U_α of X such that $V_\alpha = U_\alpha \cap Y$ for every $\alpha \in \Delta$. So $A \subseteq \cup\{V_\alpha: \alpha \in \Delta\} = \cup\{U_\alpha \cap Y: \alpha \in \Delta\} \subseteq \cup\{U_\alpha: \alpha \in \Delta\}$. Then the family $\{U_\alpha: \alpha \in \Delta\}$ is a cover of A and $U_\alpha \in P_SO(X)$. Since A is P_S-set of X, there exists a finite subset Δ_0 of Δ such that $A \subseteq$

$\cup\{U_\alpha: \alpha \in \Delta_0\}$. Since $A \subseteq Y$. Hence $A \subseteq \cup\{U_\alpha \cap Y: \alpha \in \Delta_0\} = \cup\{V_\alpha: \alpha \in \Delta_0\}$. Therefore, A is P_S-set of Y.

Sufficiency: Suppose that A is P_S-set of Y and $Y \in RO(X)$. Let $\{U_\alpha: \alpha \in \Delta\}$ be a cover of A and $U_\alpha \in P_SO(X)$ for every $\alpha \in \Delta$. Since $A \subseteq Y$. Then $A \subseteq \cup\{U_\alpha: \alpha \in \Delta\} \cap Y = \cup\{U_\alpha \cap Y: \alpha \in \Delta\}$. Since $Y \in RO(X)$. Then by Lemma 2.2 (4), there exists a P_S-open set V_α of Y such that $V_\alpha = U_\alpha \cap Y$ for every $\alpha \in \Delta$. Then the family $\{V_\alpha: \alpha \in \Delta\}$ is a cover of A and $V_\alpha \in P_SO(Y)$. Since A is P_S-set of Y, there exists a finite subset Δ_0 of Δ such that $A \subseteq \cup\{V_\alpha: \alpha \in \Delta_0\} = \cup\{U_\alpha \cap Y: \alpha \in \Delta_0\} \subseteq \cup\{U_\alpha: \alpha \in \Delta_0\}$. Therefore, A is P_S-set of X.

5 Results on Images of P_S-Compactness

Theorem 5.1: If a function $f: X \to Y$ is P_S-continuous (resp., almost P_S-continuous) and A is P_S-set, then $f(A)$ is compact (resp., N-closed) relative to Y.

Proof: Let $\{G_\alpha: \alpha \in \Delta\}$ be any cover of $f(A)$ by open sets of Y. For each $x \in A$, there exists an $\alpha(x) \in \Delta$ such that $f(x) \in G_{\alpha(x)}$. Since f is P_S-continuous (resp., almost P_S-continuous), there exists a P_S-open set U_x of X containing x such that $f(U_x) \subseteq G_{\alpha(x)}$ (resp., $f(U_x) \subseteq Int(Cl(G_{\alpha(x)}))$). Then the family $\{U_\alpha: x \in A\}$ is a P_S-open cover of A. For some finite subset A_0 of A, we have $A \subseteq \cup\{U_x: x \in A_0\}$. Therefore, $f(A) \subseteq \cup\{G_{\alpha(x)}: x \in A_0\}$ (resp., $f(A) \subseteq \cup\{Int(Cl(G_{\alpha(x)})): x \in A_0\}$). This shows that $f(A)$ is compact (resp., N-closed) relative to Y.

Corollary 5.2: If $f: X \to Y$ is P_S-continuous (resp., almost P_S-continuous) surjection function and X is P_S-compact, then Y is compact (resp., nearly compact).

Proposition 5.3: If a function $f: X \to Y$ is P_S-continuous (resp., almost P_S-continuous), A is P_S-set and F is P_S-closed subset of X, then $f(A \cap F)$ is compact (resp., N-closed) relative to Y.

Proof: Follows from Theorem 5.1 and Theorem 4.12.

Proposition 5.4: If $f: X \to Y$ is a precontinuous (resp., almost precontinuous) surjection function and X is semi-T_1 and P_S-compact space, then Y is compact (resp., nearly compact).

Proof: Follows from Theorem 5.1 and Corollary 3.19.

Proposition 5.5: If $f: X \to Y$ is a P_S-continuous (resp., almost P_S-continuous) surjection function and X is locally indiscrete and compact space, then Y is compact (resp., nearly compact).

Proof: Follows from Theorem 5.1 and Proposition 3.25.

Proposition 5.6: If $f: X \to Y$ is a continuous (resp., almost continuous) surjection function and X is locally indiscrete and P_S-compact space, then Y is compact (resp., nearly compact).

Proof: Follows from Theorem 5.1 and Proposition 3.25.

Proposition 5.7: If $f: X \to Y$ is a continuous (resp., almost continuous) surjection function and X is s-regular and P_S-compact space, then Y is compact (resp., nearly compact).

Proof: Follows from Theorem 5.1 and Lemma 3.26.

Proposition 5.8: If $f: X \to Y$ is a θs-continuous surjection function and X is extremally disconnected and P_S-compact space, then Y is compact.

Proof: Follows from Theorem 5.1 and Theorem 2.4.

Theorem 5.9: If $f: X \to Y$ is a continuous and open function. If A is P_S-set, then $f(A)$ is P_S-set.

Proof: Let $\{V_\alpha: \alpha \in \Delta\}$ be any cover of $f(A)$ by P_S-open sets of Y. Since f is continuous and open function. By Theorem 2.3, $\{f^{-1}(V_\alpha): \alpha \in \Delta\}$ is a cover of A by P_S-open sets of X. Since A is P_S-set, there exists a finite subset Δ_0 of Δ such that $A \subseteq \cup\{f^{-1}(V_\alpha): \alpha \in \Delta_0\}$. Thus, we have $f(A) \subseteq \cup\{V_\alpha: \alpha \in \Delta_0\}$. This shows that $f(A)$ is P_S-set.

Corollary 5.10: If X is a P_S-compact space and $f: X \to Y$ is a continuous and open surjection function, then Y is P_S-compact.

6 Conclusion

In this paper, we introduce P_S-compact spaces via P_S-open sets which are lies strictly between the classes of strongly compact and nearly compact spaces, but it is not comparable with compact space.

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